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EXACT SOLUTIONS FOR ELASTIC CABLE SYSTEMS

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Abstract—Analytical solutions are derived for the static response of two elastic cable systems under the action of distributed and concentrated vertical loadings. The first system is a general cable truss with an arbitrary number of vertical spacers, the second a highly symmetric 2×2 cable network. The analysis of both features a pair of lagrangian coordinates, one associated with the strained profile and the other with the unstrained profile. Cable tensions and positions in the strained state are represented as functions of the latter. A specific application of these functions is made for each system.

INTRODUCTION

The increasing structural use of cable systems—both networks and trusses—has given rise to a substantial technical literature concerning their static response, as is indicated by the report of the Subcommittee on Cable-Suspended Structures of the ASCE[1]. Largely because of the complexity of these systems the greater part of this literature is concerned with numerical approaches to the problem and in particular with finite-element methods (see, e.g. [2]). The relatively few analytical contributions have been approximate in nature, typically replacing the cable network or truss with an equivalent membrane (see [3–5] and the references contained therein). The distinguishing feature of this paper is that it proceeds from a natural formulation of an actual cable system and establishes analytical representations of its response without recourse to further simplifications—in this sense the work presented may be considered to be exact.

Determining exact solutions for elastic cable systems is a class of problems of some antiquity. *Circa* 1890, Routh[6] derived the solution for the static response of a single elastic cable under the action of self-weight alone—the so-called elastic catenary. Feld[7] generalized Routh's analysis to the unsymmetrically suspended case while Schleicher[8] treated the symmetric elastic cable with a concentrated load at its mid-point. More recently Irvine and Sinclair[9] furnished the solution for the unsymmetrically suspended elastic cable subjected to any number of concentrated loads: here in essence we extend the approach adopted in [9] and apply it to two cable systems.

The first system is an elastic cable truss comprised of two cables anchored at their ends to rigid supports and separated by any number of vertical bars which may be arbitrarily spaced horizontally. Loading of the truss is provided by the cable and bar self-weights, by distributed vertical loads along the cables, and by any number of concentrated vertical forces which may act anywhere along the upper cable. The forces within the cable truss and the associated displacements under such loadings are sought. This is a general problem of some practical significance.

In formulating the cable truss problem in Section 1, two lagrangian coordinates are used for each cable, one being the length of cable between a support and some particle point in the unstrained profile while the other is the corresponding length in the strained profile. Analytical expressions are then derived which give the forces within the truss and the rectangular coordinates describing the strained profile as functions of the lagrangian coordinate associated with the unstrained profile. These expressions contain unknowns; more precisely, for a cable truss with N spacer bars there are $4(N+1)$ unknowns. With a judicious choice of boundary conditions in the formulation, the associated $4(N+1)\times 4(N+1)$ set of equations decouples enabling the solution to be found readily, as demonstrated at the end of Section 1 by an analysis of an example entailing four spacer bars wherein the actual computations are performed on the ubiquitous pocket calculator.

The second cable system considered is a cable network. Here, however, the application of the approach used for the cable truss to a large $N \times N$ network may give rise to a set of $2N \times 2N$ equations without any decoupling. Consequently in Section 2 attention is confined to a simple symmetric 2×2 network which only requires that a pair of transcendental equations be solved simultaneously. The value of such a solution is expected to lie in checking more powerful, finite-element methods and in appraising their discretization errors. To this end the paper concludes with an application which affords a comparison with a finite-element analysis.

1. CABLE TRUSS

In formulating the cable truss problem we first consider the truss geometry and the coordinate systems required (Fig. 1). The cable truss consists of an upper cable attached at two fixed points P_0 and P_{N+1}, a lower cable attached at P'_0 and P'_{N+1}, and a set of N vertical bars connected to the upper cable at P_n and the lower cable at P'_n ($n = 1, 2, \ldots, N$).† We let $R_n(R'_n)$ refer to the cable segment contained within the points $P_n, P_{n+1}(P'_n, P'_{n+1})$. Two plane, rectangular, cartesian, coordinate systems which share a common origin at P_0 are used to describe the truss in the strained profile (i.e. when fully loaded). With respect to these systems a point P on the upper cable has coordinates (x, y), P_0 coordinates $(0, 0)$, P_n coordinates (x_n, y_n) and P_{N+1} coordinates (l, y_{N+1}); a point P' on the lower cable has coordinates (x', y'), P'_0 coordinates $(0, h_0)$, P'_n coordinates $(x_n, y_n + h_n)$ and P'_{N+1} coordinates $(l, y_{N+1} + h_{N+1})$. Thus l is the cable truss span, h_0 and h_{N+1} the vertical separations of the supports at the truss ends, and h_n the length of the bars. Central to our analysis of this elastic cable system is the introduction of two further pairs of coordinates: for P on the upper cable these are the lagrangian coordinates (p, s) where p is the length of cable between P_0 and P in the strained profile, s the length of cable in the unstrained profile; for P' on the lower cable, these are the lagrangian coordinates (p', s') defined analogously. With respect to these last systems P_0 has coordinates $(0, 0)$, P_n coordinates (p_n, s_n) and P_{N+1} coordinates $(\mathscr{L}, L)$; P'_0, P'_n, P'_{N+1} the primed counterparts. Thus $\mathscr{L}$ and L ($\mathscr{L}'$ and L') are the total upper (lower) cable lengths in the strained and unstrained states, respectively. Now since the cartesian coordinates and two of the lagrangian coordinates all refer to cables in strained profile the following relationships between the coordinate systems

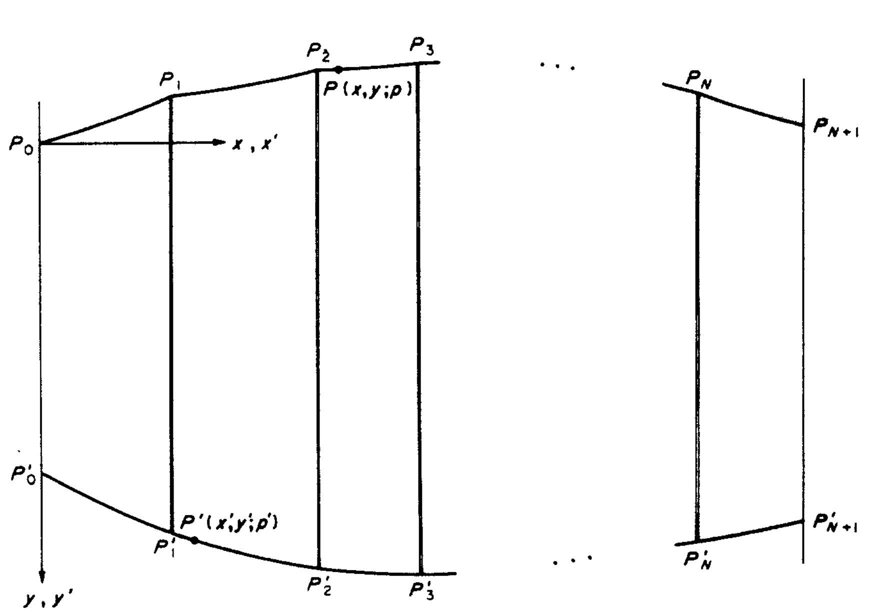

Fig. 1. Coordinates for the strained cable truss.

†Throughout we employ a prime to distinguish quantities associated with the lower cable from those with the upper cable.

must hold: for the upper cable,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2=1, \tag{1.1}$$

on $(n=0,1,\ldots,N)$, R_n, and for the lower cable a like relationship wherein the quantities are primed.

We next turn to the equilibrium requirements for the truss under the action of weight loading, distributed loading and concentrated vertical loads, focusing initially on the upper cable. We restrict distributed loadings to those which are uniform with respect to the cable length in the unstrained profile and can therefore be combined with the weight of the cable itself; accordingly we term w the effective self-weight of the upper cable per unit length (s) and let it incorporate both types of loading. Without loss of generality we assume the vertical concentrated loads F_n are applied coincident with the bars and that the portion of each load carried by the upper cable is f_n (positive in the y-direction) the remainder being transferred to the bars.† At the support P_0 a vertical reaction, f_0, is induced by these loads and acts in conjunction with an applied horizontal tension H (f_0 and H being positive in the y-, x-directions). At any point P the tension force within the cable is T. Hence resolving the forces acting on a segment of the cable between P_0 and P horizontally and vertically gives

$$T\frac{\mathrm{d}x}{\mathrm{d}p}=H,$$

$$T\frac{\mathrm{d}y}{\mathrm{d}p}=-\sum_{i=0}^{n}f_i-ws, \tag{1.2}$$

on R_n. Analogously one arrives at a pair of equilibrium requirements for the lower cable which are the same as (1.2) except that all the terms are primed. Equilibrium of the bars further requires that

$$f_n'=F_n-f_n-\bar{w}_nh_n, \tag{1.3}$$

for $n=1,2,\ldots,N$, where $\bar{w}_n$ is the bar weight per unit strained length.

In establishing constitutive relations for the cable comprising the truss we assume: the only stresses present are axial and tensile and are uniformly distributed across a section of a cable (that is, the cables are flexible), and the strains are infinitesimal. Then from the theory of elasticity we have, for the upper cable,

$$T=EA\left(\frac{\mathrm{d}p}{\mathrm{d}s}-1\right), \tag{1.4}$$

on R_n, where E is the modulus of elasticity of the upper cable and A its cross-sectional area in the unstrained profile (both constant), with, again, the attendant relation for the lower cable being obtained by merely inserting primes in (1.4).

To complete our formulation we state the various boundary conditions that are assumed to apply to the truss. The end conditions which hold at the rigid cable supports are

$$x=y=0 \quad \text{at} \quad P_0, x=l, y=y_{N+1} \quad \text{at} \quad P_{N+1},$$
$$x'=0, y'=h_0 \quad \text{at} \quad P_0', x'=l \quad \text{at} \quad P_{N+1}', \tag{1.5}$$

l, y_{N+1} and h_0 being given. The matching conditions which ensure continuity of the cables at the

†Without loss of generality because if a concentrated load is applied where there is no actual bar we regard the "bar" there as being imaginary and take $F_n=f_n$, and if no concentrated load acts at a given bar we make f_n equivalent to the reaction of the bar on the cable and set $F_n=0$.

bar attachments are

$$x_n^- = x_n^+,\ y_n^- = y_n^+ \quad \text{at} \quad P_n, \tag{1.6}$$

for $n = 1, 2, \ldots, N$, where $x_n^- = \lim_{\epsilon \to 0} x(s_n - \epsilon)$, $\epsilon > 0$, etc. with a like set of conditions for the lower cable. The horizontal spacing of the bars is assumed to be prescribed; thus

$$x = x_n \quad \text{at} \quad P_n,\ x' = x_n \quad \text{at} \quad P_n', \tag{1.7}$$

for $n = 1, 2, \ldots, N$, with x_n given. The "roof" conditions reflect the fact that typically the structure supported by the truss determines the profile of the upper cable; assuming that there are a sufficient number of points P_n to characterize the desired shape we take these as being

$$y = y_n \quad \text{at} \quad P_n, \tag{1.8}$$

for $n = 1, 2, \ldots, N$ with y_n given.† Finally, we take the vertical reaction at P_0', f_0', to be given. This last is *in lieu* of the more natural condition which sets the vertical coordinate of the support at P'_{N+1}; we postpone further examination of this exchange until the solution procedure for the truss has been developed.

Observe that any one of the coordinates x, y, p or s (x', y', p' or s') may serve as the single independent variable required for a solution for the upper (lower) cable. Implicit in the preceding formulation is the choice of $s(s')$ as this variable: we choose $s(s')$ since it gives rise to a solution that is readily applied. Accordingly we now seek the cable tensions and the rectangular coordinates of the strained cables as functions of $s(s')$. First the tensions.

Squaring the equations in (1.2) and adding the resulting expressions using (1.1) immediately provides, as the solution for the tension in the upper cable,

$$T(s) = \sqrt{\left(\left(\sum_{i=0}^{n} f_i + ws\right)^2 + H^2\right)}, \tag{1.9}$$

on R_n. Similarly the tension in the lower cable is given by (1.9) primed.

Next we treat the horizontal coordinate of the strained upper cable, x. With a view to obtaining a solution of the form $x = x(s)$, we initially seek an expression for $\mathrm{d}x/\mathrm{d}s$. Noting that $\mathrm{d}x/\mathrm{d}s = (\mathrm{d}x/\mathrm{d}p)(\mathrm{d}p/\mathrm{d}s)$ and using the first of (1.2), together with (1.4), we obtain $\mathrm{d}x/\mathrm{d}s$ in terms of T. Substituting for T from (1.9) yields an expression which may readily be integrated. The constants of integration so produced can be evaluated using the first of the end conditions in (1.5) and the first set of matching conditions in (1.6) so that one has, as the solution for the horizontal coordinate of the upper cable,

$$x(s) = \hat{c}s + \hat{w}^{-1}\left[\sinh^{-1}\left(\sum_{i=0}^{n} \hat{f}_i + \hat{w}s\right) + \sum_{i=0}^{n}\left\{\sinh^{-1}\left(\sum_{j=1}^{i} \hat{f}_{j-1} + \hat{w}s_i\right) - \sinh^{-1}\left(\sum_{j=0}^{i} \hat{f}_j + \hat{w}s_i\right)\right\}\right], \tag{1.10}$$

on R_n, where $\hat{c} = H/EA$, $\hat{w} = w/H$, $\hat{f}_i = f_i/H$, with the understanding that no contributions result from summations if the upper limit is less than the lower and that $s_0 = 0$.

A similar procedure yields the vertical coordinate of the strained upper cable,

$$y(s) = -\hat{c}s(\hat{f}_0 + \hat{w}s/2) + \hat{w}^{-1}\left[\sqrt{(\hat{f}_0^2 + 1)} - \sqrt{\left(\left(\sum_{i=0}^{n} \hat{f}_i + \hat{w}s\right)^2 + 1\right)} + \sum_{i=1}^{n}\left\{\hat{c}\hat{w}\hat{f}_i(s_i - s) + \sqrt{\left(\left(\sum_{j=0}^{i} \hat{f}_j + \hat{w}s_i\right)^2 + 1\right)} - \sqrt{\sum_{j=1}^{i}\left(\left(\hat{f}_{j-1} + \hat{w}s_i\right)^2 + 1\right)}\right\}\right], \tag{1.11}$$

†Indeed we can always ensure there are sufficient P_n by inserting additional imaginary bars/loads ($F_n = f_n = 0$).

on R_n. Inserting primes in (1.10), (1.11) and adding h_0 to the latter furnishes the expressions for the rectangular coordinates of the strained lower cable and completes our solution representations.

The results in (1.10), (1.11) and their primed companions contain a number of unknowns. For the upper cable these are the cable lengths s_{n+1} and the forces f_n ($n=0,1,\ldots,N$; $s_{N+1}=L$); for the lower cable, the cable lengths s'_{n+1} and the vertical separations h_{n+1} ($n=0,1,\ldots,N$; $s'_{N+1}=L'$). To determine these $4(N+1)$ unknowns we have available the $3(N+1)$ conditions remaining to be satisfied in (1.5), (1.7), (1.8) together with the $(N+1)$ requirements that $y'_{n+1}=y_{n+1}+h_{n+1}$ ($n=0,1,\ldots,N$). Moreover, as a result of the selection of the boundary conditions in the formulation, the entire equation system decouples into $(N+1)$ 2×2 sets for s_{n+1}, f_n of the upper cable, $(N+1)$ single equations for s'_{n+1} of the lower cable, and $(N+1)$ direct expressions for h_{n+1}. Explicitly, if $\xi=ws_{n+1}/H$, $\eta=f_n/H$, then substituting (1.10), (1.11) into the outstanding end conditions for the upper cable in (1.5), the first of (1.7), and the roof conditions (1.8), gives, as the equations for the determination of ξ and η,

$$\hat{c}\xi+\sinh^{-1}(\xi+\eta+c_1)-\sinh^{-1}(\eta+c_2)=c_3,\ \hat{c}[\xi(\xi/2+\eta+c_1)+\eta(c_2-c_3)]$$
$$+\sqrt{((\xi+\eta+c_1)^2+1)}-\sqrt{((\eta+c_2)^2+1)}=c_4, \tag{1.12}$$

where

$$c_1=\sum_{i=0}^{n-1}\hat{f}_i,\ c_2=c_1+\hat{w}s_n,$$
$$c_3=\hat{w}x_{n+1}+\sum_{i=1}^{n}\sinh^{-1}\left(\sum_{j=1}^{i}\hat{f}_{j-1}+\hat{w}s_{i-1}\right)-\sinh^{-1}\left(\sum_{j=1}^{i}\hat{f}_{j-1}+\hat{w}s_i\right),$$
$$c_4=\hat{w}y_{n+1}-\sum_{i=1}^{n}\left\{\hat{c}\hat{w}\hat{f}_{i-1}s_{i-1}+\sqrt{\left(\left(\sum_{j=1}^{i}\hat{f}_{j-1}+\hat{w}s_{i-1}\right)^2+1\right)}-\sqrt{\left(\left(\sum_{j=1}^{i}\hat{f}_{j-1}+\hat{w}s_i\right)^2+1\right)}\right\},$$

for $n=0,1,\ldots,N$, $x_{N+1}=l$: if $\xi'=w's'_{n+1}/H$, $\eta'=w'h_{n+1}/H$, then substituting (1.10)′, (1.11)′ into the outstanding end condition for the lower cable in (1.5), the second of (1.7) and the requirements that $y'_{n+1}=y_{n+1}+h_{n+1}$, gives, as the equations for the determination of ξ',

$$\hat{c}'\xi'+\sinh^{-1}(\xi'+c'_1)=c'_2, \tag{1.13}$$

where

$$c'_1=\sum_{i=0}^{n}\hat{f}'_i,$$
$$c'_2=\hat{w}'x'_{n+1}+\sum_{i=0}^{n}\left\{\sinh^{-1}\left(\sum_{j=0}^{i}\hat{f}'_j+\hat{w}'s'_i\right)-\sinh^{-1}\left(\sum_{j=1}^{i}\hat{f}'_{j-1}+\hat{w}'s'_i\right)\right\},$$

and $\hat{c}'$, $\hat{w}'$, $\hat{f}'_i$ are the primed counterparts of $\hat{c}$, $\hat{w}$, $\hat{f}_i$, for $n=0,1,\ldots,N$, $x'_{N+1}=l$, together with a set of $(N+1)$ expressions for η' in terms of known quantities once (1.12), (1.13) are solved. Although some of the equations in (1.12), (1.13) are transcendental and the remainder are nonlinear algebraic, their numerical solution is straightforward and we demonstrate this next by analyzing a simple sample truss.

The sample cable truss is a symmetric convex arrangement having four vertical supporting bars and a central tension ring (Fig. 2). The cables themselves are highly tensioned so as to support a parabolic shaped roof which exerts a distributed loading along the upper cable (incorporated into the effective self-weight w). Additional loading is provided by the weight of the cables, the supporting bars and the tension ring, this last acting like a concentrated load at the center of the truss. The symmetry of the configuration enables the analysis to be confined to the first half of the truss alone and the numerical values prescribing this portion are given in the first two columns of Table 1.

The associated equations to be solved stem from (1.12), (1.13) on inserting the values from

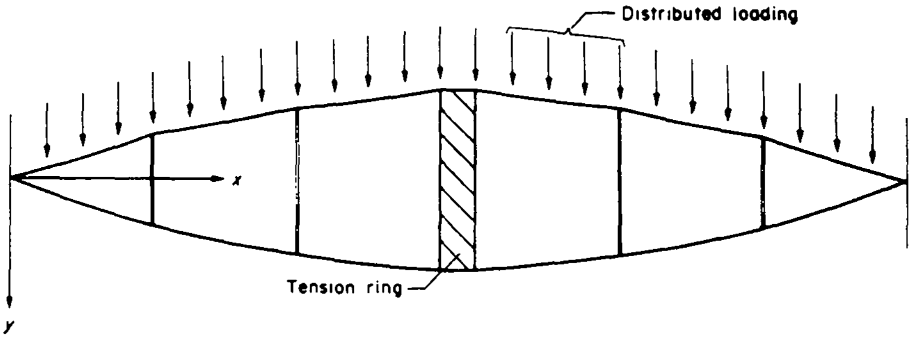

Fig. 2. Sample cable truss.

Table 1. Data and solution values for the sample cable truss

Truss geometry	Truss loading/stiffness	Unstrained upper and lower cable lengths	Vertical support reaction, bar forces/lengths
$x_1 = 10$	$w = 800$	$s_1 = 10.30$	$f_0 = 20.90$
$x_2 = 20$	$w' = \bar{w} = 17$	$s_2 = 20.40$	$f_1 = -18.16$
$x_3 = 30$	$H = 100$	$s_3 = 30.44$	$f_2 = -13.06$
$y_1 = -2.5$	$H' = 178$	$s_1' = 10.31$	$h_1 = 5.09$
$y_2 = -4.0$	$F_3 = 0.25$	$s_2' = 20.40$	$h_2 = 8.15$
$y_3 = -5.0$	$EA = 75000$	$s_3' = 30.41$	$h_3 = 9.96$

Note: x_i, y_i, s_i, s_i', h_i $(i=1,2,3)$ in feet; $w, w', \bar{w}$ in lb/ft; H, H', F_3, EA, f_i $(i=0,1,2)$ in kips.

Table 1. The order of solution is to proceed along the upper cable solving (1.12) for (s_1, f_0), then (s_2, f_1), then (s_3, f_2). With these values determined the solution for (s_1', h_1) can be addressed. For this symmetric truss, however, we need at this point to determine a value of the vertical support reaction acting on the lower cable such that the overall symmetry of the truss is preserved (see our formulation of the general truss problem wherein this reaction is presumed given). Thus we take $f_0' = -[f_0 + F_3 + ws_3 + w's_3' + \bar{w}(h_1 + h_2)]$, since though the precise values of s_3', h_1 and h_2 are not yet known, for this truss the weight forces generated by these terms are small in comparison with the external loading so that estimates of these missing values serve adequately. (While this is typically the case, in instances where it is not the solution of the equation system for the lower cable can be performed iteratively). With f_0' evaluated, (s_1', h_1), (s_2', h_2) and (s_3', h_3) are determined in turn.

A suitable approach for the actual determination of the solution in the order outlined is to use a two-dimensional Newton-Raphson method for the equation pairs arising from (1.12) coupled with a simple Newton–Raphson method for the equations from (1.13). Since the sample truss here is very taut (as is often the case in practice), a viable set of initial estimates can be obtained by rendering the truss weightless and rigid, applying a statically equivalent set of concentrated loads at the bars, then solving analytically. This procedure furnishes good initial estimates for the cable and bar lengths, no equation or equation pair taking more than three iterations to converge to four significant figures. Further, the entire process proved simple enough to be readily carried out on an HP 29C programmable calculator. The results are shown in the last two columns of Table 1.

The cable and bar lengths in Table 1 enable the truss to be "layed-out" prior to erection. The forces in Table 1 allow one to check against buckling, given the bar cross-sections (the minus signs indicate compression). All the values in Table 1 can be introduced into (1.10), (1.11) and their primed counterparts to enable the calculation of the strained profile and this is the profile drawn in Fig. 2. In the figure the small deviations from perfectly straight cable segments are consistent with the high tension in our sample truss.

It is now appropriate to review the use of the prescribed vertical support reaction instead of a condition which more directly reflects the desired shape for the truss, in particular the shape of the strained lower cable. For symmetric trusses the "preset" value must be adjusted (possible iteratively) to ensure that it equals the other vertical reaction at P'_{N+1}—in as much as this value then leads to a symmetric truss its "prescription" does shape the profile. For asymmetric trusses the advantage of fixing the vertical support reaction lies solely in the decoupling of the transcendental equations for the lower cable (see (1.13)). Nonetheless, if it is required to prescribe, say, the vertical separation of P_{N+1}, P'_{N+1}, then a shooting iterative scheme can be adopted. In general this may prove to be a slowly converging process: in actuality, for taut cable trusses, such good estimates of the right f'_0 can be found that few iterations are needed. Moreover this approach can be extended with the gross characteristics of the complete shape of the lower cable being set and both the vertical reaction f'_0 and the tension H' at P'_0 being iteratively tuned to achieve this profile.

To conclude our truss analysis some generalizations and shortcomings of the approach used bear commenting upon

—To model varying distributed loads, w can simply be made a piece-wise constant function, $w = w_n$ on R_n.

—To determine the response to additional loading (1.12) can be solved with x_n and s_n now given, and y_n and f_n as unknowns, provided the deflections are sufficiently small to be assumed vertical. However, if the added loads are large, the forces and deflections no longer remain vertical and a major generalization of our formulation to include inclined forces must be made.

2. CABLE NETWORK

In formulating the cable network problem we first consider the network geometry and the coordinate systems required (Fig. 3†). The cable network consists of four cables, joined at four intersection points P_1, P_2, P_3, P_4, with each cable attached to two fixed points at the same level, $P_5, P_6, \ldots, P_{12}$. The strained profile of the network is described by a rectangular, cartesian, coordinate system with origin P_0, y-axis positive in the downward vertical direction and a horizontal xz-plane. With respect to this system a point P on the strained profile has coordinates (x, y, z), P_0 coordinates $(0,0,0)$, P_5 coordinates $(a, 0, 0)$, P_6 coordinates $(l\text{-}a, 0, 0)$, etc. Thus a is the horizontal distance between a support and whichever of the lines $x = 0$ or l, $z = 0$ or l is closest and l is the common cable span.

When fully loaded the network is subject to the cable self-weights, uniform distributed loads and four equal, vertical, concentrated forces applied at its intersections. Under this loading it has a high degree of symmetry. The broken lines at $x = l/2$ and $z = l/2$ in Fig. 3. are both lines of symmetry. In addition the cable segments $P_{12}P_1P_{13}$ and $P_5P_1P_{14}$ are equivalent: for instance, the deflected positions of points on segment $P_{12}P_1P_{13}$ relative to the x-axis are the same as those for corresponding points on segment $P_5P_1P_{14}$ relative to the z-azis. Hence to analyze the complete network it suffices to consider only one-half of a single cable. For convenience we choose the segment $P_{12}P_1P_{13}$ in Fig. 3 and let R_1 and R_2 refer to the cable subsegments contained within the points P_{12}, P_1 and P_1, P_{13}, respectively. As in Section 1 we introduce the two lagrangian coordinates p and s for a point P on this segment. With respect to the latter P_1 is located at $s = b$, P_{13} at $s = L/2$; thus L is the total unstrained length of a cable. Similarly to Section 1 we have the following relationship between the coordinates describing the strained cable segment:

$$\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 = 1, \tag{2.1}$$

on R_n $(n = 1, 2)$.

†For clarity, the network depicted in Fig. 3 is orthogonal in plan though and analysis is not limited to this rather special case.

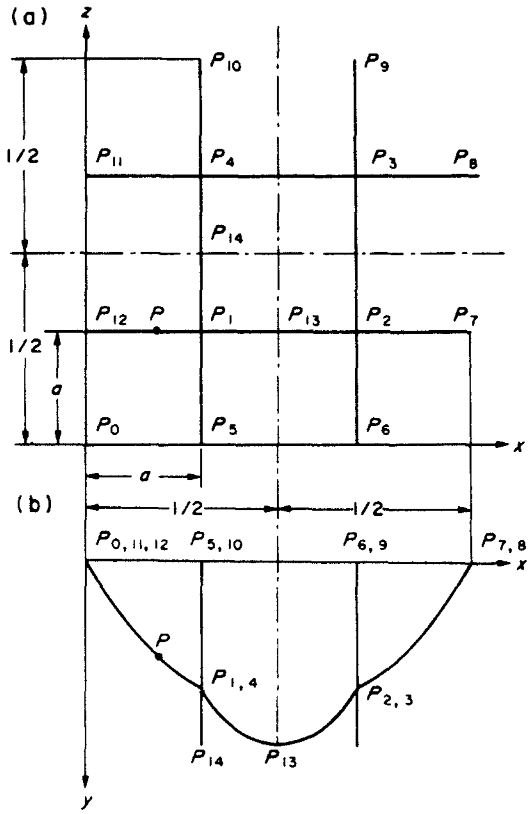

Fig. 3. Coordinates for the strained cable network: (a) Plan, (b) Elevation.

We now consider the equilibrium requirements for the selected cable segment and initially take the point P to lie in Region 1. We let F be the concentrated force applied downwards at P_1, H_x, H_z be the horizontal support reactions parallel to the x-, z-axes and T be the axial cable tension at P. Resolving in each rectangular coordinate direction and using the fact that the vertical reaction must equal $(W+F)/2$ by symmetry then gives, for Region 1,

$$T\frac{\mathrm{d}x}{\mathrm{d}p} = H_x,\ T\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{W+F}{2} - ws,\ T\frac{\mathrm{d}z}{\mathrm{d}p} = -H_z, \tag{2.2}$$

where w is the specific effective weight (effective weight per unit length s). On using the symmetries in the network, a similar procedure applied in Region 2 leads to

$$T\frac{\mathrm{d}x}{\mathrm{d}p} = H_x - H_z,\ T\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{W}{2} - ws,\ T\frac{\mathrm{d}z}{\mathrm{d}p} = 0. \tag{2.3}$$

The constitutive relation for the cable segment remains the same as in Section 1, namely as in (1.4).

To complete our formulation we state the boundary conditions that apply to the network segment. The end conditions which hold at the rigid support are

$$x = y = 0,\ z = a \quad \text{at} \quad P_{12}. \tag{2.4}$$

The matching conditions which ensure continuity of the cable segment at the intersection point are

$$x_1^- = x_1^+,\ y_1^- = y_1^+,\ z_1^- = z_1^+ \quad \text{at} \quad P_1, \tag{2.5}$$

where the minus and plus signs denote limits from Regions 1 and 2 respectively. Finally we have the conditions imposed by the symmetry of the network. The end-point P_{13} of the cable segment must lie on the line of symmetry at $x = l/2$ after loading. Further, the intersection P_1 must be on the line $z = x$, a line of symmetry not indicated in Fig. 3. Thus the two symmetry conditions are

$$x = l/2 \quad \text{at} \quad P_{13}, x = y \quad \text{at} \quad P_1. \tag{2.6}$$

Solutions for the cable segment tension and rectangular coordinates as functions of the lagrangian coordinate associated with the unstrained profile, s, follow in a completely analogous manner to those derived in Section 1. We therefore merely list the pertinent expressions here: the tensions are

$$T(s) = \sqrt{(H^2 + ((W+F)/2 - ws)^2)}, T(s) = \sqrt{((H_x - H_z)^2 + (W/2 - ws)^2)}, \tag{2.7}$$

in Regions 1, 2 respectively, where $H = \sqrt{(H_x^2 + H_z^2)}$, the resultant horizontal support reaction at P_{12}; the rectangular coordinates of the strained cable segment in Region 1 are given by

$$\begin{aligned} x(s) &= \hat{c}_x s + \hat{w}_x^{-1}[\sinh^{-1}\hat{W} - \sinh^{-1}(\hat{W} - \hat{w}s)], \\ y(s) &= \hat{c}s(\hat{W} - \hat{w}s) + \hat{w}^{-1}[\sqrt{(1+\hat{W}^2)} + \sqrt{(1+(\hat{W}-\hat{w}s)^2)}], \\ z(s) &= a - \hat{c}_z s - \hat{w}_z^{-1}[\sinh^{-1}\hat{W} - \sinh^{-1}(\hat{W} - \hat{w}s)], \end{aligned} \tag{2.8}$$

with analogous expressions for Region 2, suppressed here in the interests of brevity,† where $\hat{c}_x = H_x/EA$, $\hat{c}_z = H_z/EA$, $\hat{w}_x = W/H_x$, $\hat{w}_z = W/H_z$, $\hat{W} = (W+F)/2H$. Equations (2.8) contain unknowns. For the present formulation these are the horizontal support reactions H_x and H_z, and we now insist on the satisfaction of the two symmetry conditions to generate a pair of simultaneous transcendental equations for their determination

$$\begin{aligned} & l - \hat{c}_x L - 2w_x^{-1}[\sinh^{-1}\hat{W} - \sinh^{-1}(\hat{W} - \hat{w}b)] + \hat{c}_z(L - 2b) \\ & \quad - (\hat{w}_x^{-1} - \hat{w}_z^{-1})\sinh^{-1}(L-2b)/2(\hat{w}_x^{-1} - \hat{w}_z^{-1}) = 0, \\ & a - (\hat{c}_x + \hat{c}_z)b - (\hat{w}_x^{-1} + \hat{w}_z^{-1})[\sinh^{-1}\hat{W} - \sinh^{-1}(\hat{W} - \hat{w}b)] = 0. \end{aligned} \tag{2.9}$$

We next demonstrate how (2.9) can be solved for a specific example which may then be used to check a finite-element solution and appraise its discretization error.

As a direct check on the finite-element method developed by Saafan[2], we consider the network described in Example 2 of [2] wherein the loading consists of concentrated forces alone. The parameters for this network are: $l = 300$, $a = 100$, $L = 308.2$, $b = 104.2$ (in feet); $F = 8$, $EA = 2724$ (in kips). To check for complete agreement of Saafan's work with the analytical solution, we must render our network weightless and let $W \to 0$ in (2.8)–(2.9). With this done we solve for H_x and H_z with a two-dimensional Newton–Raphson method (to obtain reasonable initial estimates of H_x, H_z we assume that the cable material is rigid and let $EA \to \infty$) to establish $H_x = 12.69$ (kips), $H_z = 0.01681$ (kips). Equation (2.8) then gives the coordinates of the intersection as $x(b) = 99.87$ (ft), $y(b) = 31.47$ (ft). This deflected position is identical with that found by Saafan if one takes into account an initial position assumed in [2].

In order to demonstrate the evaluation of the analytical solution for a heavy cable network or a cable network under uniform distributed loading, and to estimate the discretization error in Saafan's finite-element method, we now reconsider the network defined by the parameters given in the previous paragraph with the total load divided between the concentrated forces and the cable effective weights. Hence we must solve the full set of equations in (2.9) to determine the unknown horizontal reactions and again we do this with a two-dimensional Newton–Raphson rule. The values thus found for H_x and H_y with various load distributions are exhibited in Table 2 (Col. 3, 4) together with the coordinates of a loaded intersection drawn from (2.8) (Cols. 5, 6).

†See [10] for details.

Table 2. Network response under varying load distributions

Load distribution		Values from exact analysis				Values from finite-element analysis			
F	W	H_x	H_z	$x(=z)$ at P_1	y at P_1	H_x	H_z	$x(=z)$ at P_1	y at P_1
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
7	1	12.20	0.01487	99.88	31.36	12.20	0.01503	99.88	31.37
6	2	11.71	0.01266	99.89	31.22	11.69	0.01334	99.89	31.28
5	3	11.24	0.01013	99.91	31.04	11.19	0.01174	99.90	31.18
4	4	10.77	0.00723	99.93	30.81	10.68	0.01021	99.90	31.08
3	5	10.31	0.00390	99.96	30.51	10.17	0.00878	99.91	30.98
2	6	9.88	0.00008	100.00	30.14	9.66	0.00743	99.92	30.88
1	7	9.46	-0.00428	100.05	29.66	9.14	0.00617	99.93	30.78
0	8	9.06	-0.00923	100.10	29.06	8.62	0.00500	99.94	30.68

Note: F, W, H_x, H_y in kips; x, y, z in feet.

In Saafan's finite-element method, any distributed loading is replaced by concentrated forces at element ends. Consequently, if the sample network is modeled by 12 elements, we apply "equivalent" concentrated forces of $F + W(1 - b/L)$ at the intersections of a weightless cable network. Table 2 also exhibits the values of the horizontal reactions and the coordinates of a loaded intersection under such "equivalent" loading (Cols. 7–10).

The differences between the analytical and finite-element values in Table 3 can be attributed to the discretization error in the finite-element method. This discretization error is highest when all the applied load is distributed, with the error in the resultant, horizontal, support reaction, H, being 4.9%, while the maximum error in the coordinates of the loaded intersection occurs in the vertical coordinate, y, and is 5.6%. These errors could be reduced by introducing more elements (see [10] for a fuller discussion).

In conclusion we consider the possibility of exact treatment of more complex cable networks. Extension of our network analysis to a symmetric $N \times N$ network is straightforward but tedious, giving rise to $\langle (N+1)/2 \rangle s$-$p$ coordinate pairs with an attendant set of twice as many transcendental equations.† However, since these equations do not decouple, the determination of the support reactions become more difficult. The removal of any symmetry also complicates the analysis. Consequently it would appear that the analytical approach presented here is limited to simple cable networks.

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†$\langle\ \rangle$ denotes the greatest integer part.